Feynman graphs for non-Gaussian measures

Sidi Hamidou Djah[1], Hanno Gottschalk[2] and Habib Ouerdiane[1]

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[1]: Département des Mathématiques, Université de Tunis El Manar
[2]: Institut für angewandte Mathematik, Rheinische Fridrich-Wilhelms-Universität Bonn

Abstract. Partition and moment functions for a general (not necessarily Gaussian) functional measure that is perturbed by a Gibbs factor are calculated using generalized Feynman graphs. From the graphical calculus, that is formulated using the theory of species, a new notion of Wick ordering arises, that coincides with orthogonal decompositions of Wiener-Itô type only if the measure is Gaussian. Proving a generalized linked cluster theorem, we show that the logarithm of the partition function can be expanded in terms of connected Feynman graphs.

Key words: *Generalized Feynman graphs, Wick ordering, free energy density.*
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1 Introduction

Perturbative expansions for Path integrals – both oscillatory [5, 1] and probabilistic [14, 11, 16, 6] are one of the main technical tools in contemporary elementary particle physics and statistical physics.

Expansions in terms of Feynman graphs are done starting of with a free measure of Gaussian type which is then perturbed with a local and polynomial interaction, e.g. the ϕ^4 interaction.

In this note we give some results on the perturbation theory of non-Gaussian measures of Euclidean type. The focus is on the combinatorial expansion in terms of a generalized class of graphs, henceforth called generalized Feynman graphs.

Non Gaussian measures as a starting point of the theory are of interest in statistical physics, where e.g. measures of poisson type play a great rôle [15], but recently also have attracted some attention in particle physics, cf. e.g. [13], and – properly generalized to Grassmann algebras – in the theory of solid states [3]. Physically motivated questions related to infra-red and self energy problems of systems of classical particles in the continuum can be found in [4]. Also in stochastics, related graph expansions proved to be useful in the asymptotic expansion of densities of Lévy laws [9] and the perturbation theory of Lévy driven SPDEs [8].

In particular, a reformulation of the Wilson-Polchinski renormalization group for measures other than Gaussian seems to be highly desirable, since only if different systems of statistical

mechanics can be treated on the same mathematical footing, universality can be properly understood, cf. [7] for a first step in that direction. It is clear, that all these works heavily rely on combinatorial structures as a point of departure.

In previous articles on the subject, cf. [4, 7], the Feynman graph calculus was however used as in theoretical physics, as an *ad hoc* formalism to order certain combinatorial sums. In this work we refine the definition of the main cass of objects – the generalized Feynman graphs – using the proper combinatorial tools given by the theory of species [2].

After a short section fixing the notations and briefly surveying the perturbation expansion, we introduce these generalized Feynman graphs in section 3. In section 4 we give a graphical notion of Wick ordering for general functional measures and we compare it with orthogonal expansions of Wiener-Itô-Segal type [12, 16, 6]. In section 5 we prove the linked cluster theorem for the generalized Feynman graphs, which paves the way for applications in statistical physics.

2   Perturbation theory

Let us consider the nuclear triplet

$$\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \tag{1}$$

where $d \in \mathbb{N}$, $\mathcal{S}(\mathbb{R}^d)$ the space of functions defined on $\mathbb{R}^d$, differentiable for any order, rapidly decreasing and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, i.e the topological dual of $\mathcal{S}(\mathbb{R}^d)$. We denote by $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$ the Borel $\sigma-$algebra generated by the open sets of the weak topology on $\mathcal{S}'(\mathbb{R}^d)$.

We consider functional measures ν_0 on the measurable space $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)))$ such that all moments of ν_0 exist. Let $\chi_\epsilon \in \mathcal{S}(\mathbb{R}^d)$ be a family of positive, reflection invariant function such that $\int_{\mathbb{R}^d} \chi_\epsilon(x)dx = 1$, $\chi_\epsilon \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ for $\epsilon \downarrow 0$. The function χ_ϵ can be seen as the integral kernel of an operator :

$$\chi_\epsilon : \ \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \ , \ \ \phi \mapsto \phi_\epsilon = \chi_\epsilon * \phi \tag{2}$$

where $(\chi_\epsilon * \phi)(x) = \langle \phi, \chi_{\epsilon,x} \rangle$, $\chi_{\epsilon,x}(y) = \chi_\epsilon(y - x)$.

The convolution $\chi_\epsilon * \phi$ is continuous in the weak $\mathcal{S}'(\mathbb{R}^d)$ topology and in particular it is $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d)) - \mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$ measurable. It takes values in the $C^\infty(\mathbb{R}^d)-$functions with tempered (at most polynomial) increase at infinity, $C^\infty_{temp}(\mathbb{R}^d)$.

We consider the functional measure $\nu_{0,\epsilon}$ which is the image measure of ν_0 under $\chi_\epsilon : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. In particular, the support of $\nu_{0,\epsilon}$ lies in the image of $\mathcal{S}'(\mathbb{R}^d)$ under χ_ϵ, hence in the space of $C^\infty_{temp}(\mathbb{R}^d)$. It is thus possible, to approximate (in law) any measure ν_0 by measures $\nu_{0,\epsilon}$ which have support not only on tempered distributions, but on $C^\infty_{temp}(\mathbb{R}^d)-$functions. This goes under the name ultra-violet (uv) regularization procedure.

From now on we assume that ν_0 is suitably uv-regularized, i.e. has support on $C^\infty_{temp}(\mathbb{R}^d)$. As a measurable space, $C^\infty_{temp}(\mathbb{R}^d)$ is equipped with the trace σ algebra $\mathcal{B} = C^\infty_{temp}(\mathbb{R}^d) \bigcap \mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$. We can note that for $x \in \mathbb{R}^d$, the mapping $C^\infty_{temp}(\mathbb{R}^d) \ni \phi \mapsto \phi(x) \in \mathbb{R}$ is measurable as the pointwise limit of $\langle \phi, \chi_{\epsilon,x} \rangle \to \phi(x)$ $\epsilon \downarrow 0$. Having specified the conditions on ν_0, we next define interactions of local type :

Let $v : \mathbb{R} \to \mathbb{R}$ be a function which is continuous and bounded from below. Let Λ be a compact subset of $\mathbb{R}^d$. Then we set

$$V_\Lambda(\phi) = \int_\Lambda v(\phi(x))dx, \ \ \phi \in C^\infty_{temp}(\mathbb{R}^d) \tag{3}$$

We are particularly interested in the case where v is a monomial and we chose $v(\phi) = \phi^4$ for the sake of concreteness following the tradition in perturbation theory. Furthermore, it is easy to show that $V_\Lambda : C^\infty_{temp}(\mathbb{R}^d) \to \mathbb{R}$ is $\mathcal{B}-$measurable and bounded from below. Hence, $V_\Lambda, e^{-\lambda V_\Lambda} \in L^p(\nu_0)$ for $\lambda > 0, p \geq 1$ by the assumption on existence of moments and semiboundedness. Let ν_Λ^λ be the non-normalized perturbed measure defined by :

$$d\nu_\Lambda^\lambda(\phi) = e^{-\lambda V_\Lambda(\phi)} d\nu_0(\phi) \tag{4}$$

where $0 < \lambda \ll 1$. λ is called the coupling constant.
We want to calculate the moments of ν_Λ^λ :

$$S_{n,\Lambda}^\lambda(x_1, ..., x_n) = \int_{C^\infty_{temp}} \phi(x_1)...\phi(x_n) e^{-\lambda V_\Lambda(\phi)} d\nu_0(\phi) \tag{5}$$

Next we come to the perturbation series of the expression (5). One can expand the exponential in (5) in powers of the coupling constant :

$$S_{n,\Lambda}^\lambda(x_1, ..., x_n) = \int_{C^\infty_{temp}} \phi(x_1)...\phi(x_n) \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!} \left[\int_\Lambda \phi^4(x) dx \right]^m d\nu_0(\phi) \tag{6}$$

Next one interchanges the infinite sum and the integral w.r.t ν_0. This is in general only a formal operation :

$$S_{n,\Lambda}^\lambda(x_1, ..., x_n) = \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!} \int_{C^\infty_{temp}} \phi(x_1)...\phi(x_n) \left[\int_\Lambda \phi^4(x) dx \right]^m d\nu_0(\phi) \tag{7}$$

Applying Fubini's lemma to the right hand side of (7) one obtains

$$S_{n,\Lambda}^\lambda(x_1, ..., x_n) = \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!} \int_{\Lambda^m} \int_{C^\infty_{temp}} \phi(x_1)...\phi(x_n)\phi^4(y_1)...\phi^4(y_m) d\nu_0(\phi) dy_1...dy_m \tag{8}$$

The problem with (8) is that for a number of interesting examples the series diverges, while (6) is well-defined. From the semiboundedness of V_Λ one obtains that $S_{n,\Lambda}^\lambda$ is C^∞ in λ on $[0, +\infty)$ however not necessarily analytic at 0, [4, Lemma 2.2]. Therefore, a finite partial sum up to the order $N \in \mathbb{N}$ in the coupling constant $\lambda > 0$ of the divergent series (8) for λ small gives an excellent approximation of (6), i.e. for all $N \in \mathbb{N}$ fixed :

$$\left| S_{n,\Lambda}^\lambda(x_1, ..., x_n) - \sum_{m=0}^N \frac{(-\lambda)^m}{m!} \int_{\Lambda^m} \int_{C^\infty_{temp}} \phi(x_1)...\phi(x_n)\phi^4(y_1)...\phi^4(y_m) d\nu_0(\phi) dy_1...dy_m \right| < c_N \lambda^{N+1} \tag{9}$$

for some $c_N > 0$ by Taylor's lemma. Note that in particular the case $n = 0$ gives us an expression for the sum over states

$$S_{0,\Lambda}^\lambda = \Xi_\Lambda(\lambda) = \int_{C^\infty_{temp}} e^{-\lambda V_\Lambda(\phi)} d\nu_0(\phi) \tag{10}$$

and the moment functions for the normalized measure $\Xi_\Lambda(\lambda)^{-1}\nu_\Lambda^\lambda$ can be obtained from the expansion on $S_{n,\Lambda}^\lambda$ and $\Xi_\Lambda(\lambda)$ by the procedure of inversion and multiplication of inverse power

series. In section 5 we will also consider the case of the free energy $\ln \Xi_\Lambda(\lambda)$ which is the main object of interest in thermodynamics.

To obtain the formal perturbation series, it is enough to evaluate the integrals in the formal power series (8) :

$$\int_{\Lambda^m} \int_{C_{temp}^\infty} \phi(x_1)...\phi(x_n)\phi^4(y_1)...\phi^4(y_m)d\nu_0(\phi)dy_1...dy_m$$

$$= \int_{\Lambda^m} \langle\phi(x_1)...\phi(x_n)\phi^4(y_1)...\phi^4(y_m)\rangle_{\nu_0} dy_1...dy_m \tag{11}$$

In the following section we give a graphical method for the evaluation of such integrals. This method is standard for the special case when ν_0 is Gaussian and then gives the well-known Feymann diagrams and rules. The generalization to non-Gaussian measures is our objective.

3 Generalized Feynman graphs

The content of this and the following sections is essentially combinatorial. Therefore give a short digression on the theory of species [2] which provides the right language to deal with the partially labeled objects that are needed in the following.

Definition 3.1. *A species of structures is a rule F such that for any finite set U another finite set $F[U]$ is generated and furthermore for any bijection $\sigma : U \to V$ of finite sets a mapping $F[\sigma] : F[U] \to F[V]$ is generated fulfilling the following functorial properties*

(i) $F[I_U] = I_{F[U]}$ where I stands for the identity mapping;

(ii) $F[\sigma] \circ F[\tau] = F[\sigma \circ \tau]$, for $U \xrightarrow{\tau} V \xrightarrow{\sigma} W$ bijections.

Note that properties (i) and (ii) together imply that $F[\sigma]$ is a bijection, if σ is.

Let us give some useful examples: For any set U we may set $I[U] = U$ and $I[\sigma] = \sigma$ and we trivially get a species. Another species is the species of partitions $\mathcal{P}[U] = \{I = \{I_1, \ldots, I_k\} : k \in \mathbb{N}^*, I_j \subseteq U, I_l \cap I_j = \emptyset$ for $j \neq l, j,l = 1, \ldots, k\}$ with $\mathcal{P}[\sigma](I) = \{\sigma(I_1), \ldots, \sigma(I_k)\}$ if $I = \{I_1, \ldots, I_k\}$.

For later use we note that the notion of species over one finite set U immediately generalizes zu ther notion of species over an ordered pair of sets (U,V) with isomorphisms $\sigma = (\sigma_1, \sigma_2) : (U,V) \to (W,Y)$ such that $\sigma_1 : U \to W$ and $\sigma_2 : V \to Y$ are bijections of finite sets. We also introduce the notation $[n]$ for the set $\{1, \ldots, n\}$ and nU for $U \times [n]$.

We now come back to the point that in the formula (8) in Section 2, the calculation of the perturbation series can be reduced to the calculation of the moments (11) of the unperturbed measure ν_0. But often it is easier to calculate truncated moments (also called truncated Schwinger functions) which are defined as follows:

Definition 3.2. *Let ν_0 be a probability measure defined on some measurable space and let $X_1, X_2, \ldots$ be (not necessarily distinct) random variables on that probability space that are L^p integrable for $p \geq 1$. Then $\{\langle X_{j_1} \cdots X_{j_n}\rangle_{\nu_0}\}$, $n \in \mathbb{N}^*$, $j_1 \ldots, j_n \in \mathbb{N}$ is the associated collection*

of moments. The truncated sequence of monemts, $\langle X_{j_1}, \ldots, X_{j_n} \rangle_{\nu_0}^T$ is then defined recursively via

$$\langle X_{j_1} \cdots X_{j_n} \rangle_{\nu_0} = \sum_{\substack{I \in \mathcal{P}(\{1,\ldots,n\}) \\ I = \{I_1,\ldots,I_k\}}} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T \tag{12}$$

here $\langle I_l \rangle_{\nu_0}^T = \langle X_{j_{i_1}}, \ldots, X_{j_{i_q}} \rangle_{\nu_0}^T$ for $I_l = \{i_1, \ldots, i_q\}$.

In particular let ν_0 be a measure on some space of functions and $X_j = \phi(u_j)$ for $u_1, u_2 \ldots \in \mathbb{R}^d$. The truncated moments $\langle \phi(u_1), \ldots, \phi(u_n) \rangle_{\nu_0}^T$ of ν_0 are also called the truncated Schwinger functions of ν_0.

The following theorem is well-known:

Theorem 3.3. *(Linked Cluster theorem)*
Let $C(t_1, u_1, \ldots, t_n, u_n)$, $n \in \mathbb{N}$, $u_j \in \mathbb{R}^d$, $t_j \in \mathbb{R}$ be the Fourier transform of ν_0, i.e :

$$C(t_1, u_1, \ldots, t_n, u_n) = \int e^{i \sum_{j=1}^{n} t_j \phi(u_j)} d\nu_0(\phi) = \hat{\nu_0}\left(\sum_{j=1}^{n} t_j \delta_{u_j}\right). \tag{13}$$

Then the moments are generated by C :

$$\langle \phi(u_1) \ldots \phi(u_n) \rangle_{\nu_0} = (-i)^n \frac{d^n}{dt_1 \ldots dt_n} C(t_1, u_1, \ldots, t_n, u_n)|_{t_1 = \ldots t_n = 0}. \tag{14}$$

Likewise, let $C^T(t_1, u_1, \ldots, t_n, u_n) = \ln C(t_1, u_1, \ldots, t_n, u_n)$ which is well-defined for $t_1, \ldots, t_n$ sufficiently small. Then C^T generates the truncated moments :

$$\langle \phi(u_1), \ldots, \phi(u_n) \rangle_{\nu_0}^T = (-i)^n \frac{d^n}{dt_1 \ldots dt_n} C^T(t_1, u_1, \ldots, t_n, u_n)|_{t_1 = \ldots t_n = 0} \tag{15}$$

Example 3.4. (The Gaussian case) If ν_0 is Gaussian measure with covariance $G(u, v)$ and mean 0, then :

$$C(t_1, u_1, \ldots, t_n, u_n) = e^{-\frac{1}{2} \sum_{j,\,l=1}^{n} t_j t_l G(u_j, u_l)}$$

and consequently

$$\langle \phi(u_1) \phi(u_2) \rangle_{\nu_0} = G(u_1, u_2), \tag{16}$$

$$C^T(t_1, u_1, \ldots, t_n, u_n) = -\frac{1}{2} \sum_{j,\,l=1}^{n} t_j t_l G(u_j, u_l) \tag{17}$$

Inserting this into (15) yields :

$$\langle \phi(u_1), \ldots, \phi(u_n) \rangle_{\nu_0}^T = \begin{cases} 0, & n \neq 2 \\ \\ G(u_1, u_2), & n = 2, \end{cases} \tag{18}$$

Using the linked Cluster theorem for this special case, one obtains from (12):

$$\langle \phi(u_1)...\phi(u_n)\rangle_{\nu_0} = \sum_{I\in\mathcal{P}_e(\{1,...,n\})} \prod_{\{j,\,l\}\in I} G(u_j,u_l), \qquad (19)$$

where $\mathcal{P}_e(\{1,...,n\})$ is the set of all partitions of $\{1,...,n\}$ given by $I = \{I_1,...,I_{\frac{n}{2}}\}$ such that $|I_l| = 2$ if n is even. Note that $\mathcal{P}_e(\{1,...,n\}) = \emptyset$ if n is odd. Although known before [10], the above formula in the physics context is known as Wick's theorem.

Now one can use the combinatorics of truncation to obtain an expression of (11) in terms of the usually simpler truncated moments. In fact by Definition 3.2, (11) is given by :

$$\int_{\Lambda^m} \langle \phi(x_1)...\phi(x_n)\phi^4(y_1)...\phi^4(y_m)\rangle_{\nu_0} dy_1...dy_m = \sum_{I\in\mathcal{P}([n]+4[m])} \int_{\Lambda^m} \prod_{I_l\in I} \langle I_l\rangle_{\nu_0}^T dy_1...dy_m, \qquad (20)$$

where $\langle I_l\rangle_{\nu_0}^T = \langle \phi(z_{j_1}),...,\phi(z_{j_q})\rangle_{\nu_0}^T$ with $z_{j_l} = x_{j_l}$ for $j_l \in [n]$ and $z_{j_l} = y_s$ for $j_l \in 4[m]$, $j_l = (s,q)$, $s = 1,...,m$, $q = 1,...,4$, $I_l = \{j_1,...j_q\}$. In fact, given finite sets U,V we could now define $F[U,V] = \mathcal{P}(U\dot{\cup}4V)$ and *define* this as the species of generalized Feynman graphs with exterior (full) vertices labeled by U and interaction (inner full) vertices labeled by V. For $|U| = n$ and $|V| = m$, one can then define an evaluation mapping $\hat{\mathcal{V}}_\Lambda(I)(x_1,...,x_n)$ for $I \in F(U,V)$ by the integral on the right hand side of (20). In fact, as our calculations in the subsequent sections will show, this would be the best definition for proving theorems. It is however a bad definition for finding theoems, since this definition is not very intuitive.

To visualize the combinatorics one thus uses a representation through graphs, which is more intuitive. Now there are two problems. Firstly, what does "a representation through graphs" mean? Secondly, the standard notion of graphs has labeled vertices and non labeled "legs" (endpoints where the edges meet the vertex). This is adequate to formulate a graphic representation of the Meyer series for the low activity expansion in the staistical mechanics of gases and liquids [15]. It is however inadequate for the Feynman, high temperature expansion. It will turn out that there we need vertices which are labeled and have labeled "legs" as well as vertices which are unlabeled and have unlabeled "legs" where the term "leg" stands for the points where the edge meets the vertex.

We start with the second problem. Let U be a finite set, then $\mathcal{G}[U]$ is the species of simple graphs over the set of vertices U, i.e. $\mathcal{G}[U] = \{G \subseteq \{\{a,b\} \subseteq U : a \neq b\}\}$. The elements $e \in G$ are called edges of G. The transport along a bijection $U \xrightarrow{\sigma} V$ is defined by $\mathcal{G}[\sigma](G) = \{\sigma(e) : e \in G\}$. For a vertex $u \in U$ and a graph $G \in \mathcal{G}[U]$ the degree is defined by $f(u,G) = |\{e \in G : u \in e\}|$.

For $K \subseteq U$, let $\Sigma(K)$ be the group of permutations of K that acts as the identity on $U \setminus K$. Let $\mathcal{G}' \subseteq \mathcal{G}[U]$ be a set of simple graphs that is invariant under all $\mathcal{G}[\sigma]$ with $\sigma \in \Sigma(K)$. Then let $\mathcal{G}'/\Sigma(K)$ be the set of equivalence classes of graphs in $\mathcal{G}'$ that are being mapped onto each other via the action of $\mathcal{G}[\Sigma(K)]$. The idea is to make the vertices in K unlabeled while the vertices in $U \setminus K$ remain labeled.

Definition 3.5. *A generalized Feynman graph G of ϕ^4-theory with a set U outer full vertices, a set V of inner full vertices and a set K of inner empty vertices is a graph $G \in \mathcal{G}[U\dot{\cup}4V\dot{\cup}K]/\Sigma[K]$ such that the following conditions hold:*

   (i) *For $v \in U\dot{\cup}4V$, $f(v,\hat{G}) = 1$ and, for $v \in K$, $f(v,\hat{G}) > 0$ where $\hat{G}$ is a representative for the equivalence class G;*
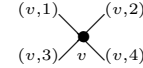
(ii) If $\hat{G}$ represents the equivalence class G, then $\forall e \in \hat{G}$, $e = \{a, b\}$ with $a \in U \dot{\cup} 4I$, $b \in K$.

Denote the set of such graphs by $\mathcal{F}(U, V, K)$. The set of all Feynman graphs with outer full vertices U and inner full vertices V is defined as $\mathcal{F}(U, V) = \dot{\cup}_{k \geq 0} \mathcal{F}(U, V, [k])$.

Next we work out a graphical representation. We use the conventions:

$$\textit{Full inner vertex} : \bullet \quad \textit{Empty inner vertex} : \circ \quad \textit{Full outer vertex} : \times \qquad (21)$$
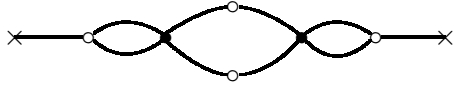
Note that in a ϕ^4 theory, each inner full vertex is of fertility four. A vertex $v \in V$ is therefore identified with the c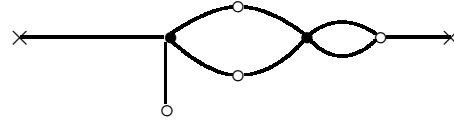ollection of its legs, $(v, 1), \ldots, (v, 4) \in 4V$: By condition (i), each leg is connected with exactly one edge. By condition (ii) full vertices are connected with empty vertices and vice versa, cf. the following examples:

Example 3.6.

Generalized Feynman Graph   No *Generalized Feynman Graph*

Figure 1

The figure on right in Example 3.6 is no Generalized Feynman graph, because there is an edge which connects two full vertices. As it is customary, only the topological graph (i.e. without labelings) is displayed.

We now come back to the first of the above mentioned problems. In what sense $F(U, V)$ and $\mathcal{F}(U, V)$ can be identified? The following definition helps:

Definition 3.7. *Two species H and R are equivalent, if for each finite set U there is a bijection $\alpha[U] : H[U] \to R[U]$ such that for any finite set V and any bijection $\sigma : U \to V$, the following diagram commutes*

$$\begin{array}{ccc} H[U] & \xrightarrow{\alpha[U]} & R[U] \\ {\scriptstyle H[\sigma]} \downarrow & & \downarrow {\scriptstyle R[\sigma]} \\ H[V] & \xrightarrow{\alpha[V]} & R[V] \end{array} \qquad (22)$$

Again, the generalization to species that depend on two finite sets is immediate. We thus have to construct the bijections $\alpha[U, V] : \mathcal{F}[U, V] \to F[U, V]$, which is done as follows: Suppose that $G \in \mathcal{F}[U, V]$ has k inner empty vertices. Label these vertices with $1, \ldots, k$. Then define I_l as the set of legs in $4V$ that are connected with the l-th inner empty vertex, together with the outer full vertices in U that are connected with the l-th inner full vertex. Then $I = \{I_1, \ldots, I_k\} \in \mathcal{P}(U \dot{\cup} 4V) = F[U, V]$ obviously does not depend on the labeling of the inner empty vertices.

The construction of the inverse mapping, $\alpha^{-1}[U, V]$ is most easily understood in the case of an example, which generalizes in a straightforward manner to the general case: Let us consider a partition that corresponds to a 2-point moment function in second order perturbation theory, i.e. $|U| = |V| = 2$. To express the moment function in the given order of perturbation theory through truncated moment functions, see Definition 3.2, we have to sum over all partitions of $2 \times 1 + 2 \times 4 = 10$ objects, as there are two exterior vertice $U = \{u_1, u_2\}$ and two interior full vertices $V = \{v_1, v_2\}$, each repeated four times since we consider a partition of $U \dot{\cup} 4V$. Take for example the following partition $I = \{I_1, I_2, I_3, I_4\}$:

Figure 2

Each element I_l, $1 \le l \le 4$ in the partition I, is now mapped to a vertex of empty type, take e.g. I_1 $= u_1 \times\!\!-\!\!-\!\!\bigcirc\!\!\!<\!\!\!>\!\bullet\, v_1$. Consequently, if there are k sets in the parti-

tion (here $k = 4$) we obtain k inner empty vertices. Label them with $1, \ldots, k$. The edges of the associated graph are then given by $\cup_{l=1}^{k}\{\{p, l\}, p \in I_l\}$. Thus, in the case of our example, $\{\{u_1, 1\}, \{(v_1, 1), 1\}, \{(v_1, 3), 1\}, \{(v_1, 2), 2\}, \{(v_1, 4), 3\}, \{(v_2, 1), 2\}, \{(v_2, 3), 3\}, \{(v_2, 2), 4\}, \{(v_2, 4), 4\}, \{u_2, 4\}\}$. The generalized Feynman graph is then the $\Sigma([k])$-equivalence class of the simpe graph defined by the list of edges, which removes the labeling $1, \ldots, k$. The above partition is thus mapped to the the following generalized Feynman graph:

Figure 3

It is easily checked that the bijections $\alpha[U, V]$ fulfill the natralness condition (22). Thus we have derived the following result:

Theorem 3.8. *The species $\mathcal{F}$ and F are equivalent.*

In particular this means that we can replace a combinatorial sum in (20) over $F(n, m) = F[[n], [m]]$ by a combinatorial sum over the generalized Feynman graphs $\mathcal{F}(n, m) = \mathcal{F}[[n], [m]]$, provided that we assign the same value to the generalized Feynman graph that has been assigned to the corresponding (under α) partition.

This brings us to the issue of Feynman rules. For $G \in \mathcal{F}(n, m)$ clearly we have to define $\mathcal{V}[G](x_1, \ldots, x_n) = \hat{\mathcal{V}}[\alpha^{-1}[[n], [m]](G)](x_1, \ldots, x_n)$. It is however not necessary to go over the detour of partitions, as the value can be read off the graph directly. This is done by an algorithm called Feynman rules. In the given example, the Feynman rules are applied as follows:

$I_1 : u_1 \times\!\!-\!\!-\!\!\bigcirc\!\!\!<\!\!\!>\!\bullet\, v_1 \leftrightarrow \langle \phi(x_1), \phi(y_1), \phi(y_1) \rangle_{\nu 0}^{T} = \langle I_1 \rangle_{\nu 0}^{T}$

$I_2 : v_1 \bullet\!\!-\!\!-\!\!\bigcirc\!\!-\!\!-\!\!\bullet\, v_2 \leftrightarrow \langle \phi(y_1), \phi(y_2) \rangle_{\nu 0}^{T} = \langle I_2 \rangle_{\nu 0}^{T}$

$I_3 : v_1 \bullet\!\!-\!\!-\!\!\bigcirc\!\!-\!\!-\!\!\bullet\, v_2 \leftrightarrow \langle \phi(y_1), \phi(y_2) \rangle_{\nu 0}^{T} = \langle I_3 \rangle_{\nu 0}^{T}$

$I_4 : u_2 \times\!\!-\!\!-\!\!\bigcirc\!\!\!<\!\!\!>\!\bullet\, v_2 \leftrightarrow \langle \phi(x_2), \phi(y_2), \phi(y_2) \rangle_{\nu 0}^{T} = \langle I_4 \rangle_{\nu 0}^{T}.$

Hence,

$$\mathcal{V}(G) = \int_\Lambda \int_\Lambda \langle \phi(x_1), \phi(y_1), \phi(y_1) \rangle_{\nu_0}^T \left(\langle \phi(y_1), \phi(y_2) \rangle_{\nu_0}^T \right)^2 \langle \phi(y_2), \phi(y_2), \phi(x_2) \rangle_{\nu_0}^T dy_1 dy_2 \qquad (23)$$

This can be generalized as follows: Associate to each inner full vertex from $[m]$ an inegration variable $y_1, \ldots, y_m$. To the outer full vertices in $[n]$, the values $x_1, \ldots, x_n$ are assigned. The connection between graphs and moments of functional measures is established through identification

$$\begin{array}{c} w_2 \\ \diagdown \quad \cdots \\ w_1 \rule[0.5ex]{3em}{0.4pt}\circ\rule[0.5ex]{3em}{0.4pt} w_q \end{array} \leftrightarrow \langle \phi(z_1), \cdots, \phi(z_q) \rangle_{\nu_0}^T \qquad (24)$$

where $w_1, \ldots, w_q \in [n] + 4[m]$ and $z_1, \ldots z_q$ are the integration/external variables associated to the full vertices to which $w_1, \ldots, w_q$ belong. Namely , each empty inner vertex in a generalized Feynman graph, with q edges connected to the legs of the full vertices labeled by $w_1, ..., w_q$, is evaluated by the truncated moment $\langle \phi(z_1) \cdots \phi(z_q) \rangle_{\nu_0}^T$. Following this rule we obtain the general description of the evaluation of generalized Feynman graphs:

Definition 3.9. *Let $x_1, .., x_n \in \mathbb{R}^d$ be values assigned to the full outer vertices. Assign intrgration variables $y_1, ..., y_m \in \mathbb{R}^d$ to the m full inner vertices. For a generalized Feynman graphs $G \in \mathcal{F}(n, m)$, one proceeds as follows :*

1. *For any empty inner vertex $\circ$ in G with q legs (edges), we associate to this empty inner vertex $\circ$, a q-point truncated moment function with arguments given by the $x_l's$ and $y_j's$ corresponding to the outer and inner full vertices that are connected by an edge to that empty vertex;*

2. *Multiply all the truncated moment functions obtained in this fashion;*

3. *Integrate this product w.r.t each variable y_j (interaction vertex) appearing in this $q-$point truncated moment function over Λ.*

Denote by $\mathcal{V}(G)$ the obtained value.

By combination of the above definition and Theorem 3.8, one obtains :

$$\int_{\Lambda^m} \int_{C_{temp}^\infty} \phi(x_1)...\phi(x_n)\phi^4(y_1)...\phi^4(y_m) d\nu_0(\phi) dy_1...dy_m = \sum_{G \in \mathcal{F}(n,m)} \mathcal{V}(G). \qquad (25)$$

Theorem 3.10. *The perturbation series (8) can be expressed as follows:*

$$S_{n,\Lambda}^\lambda(x_1, \ldots, x_n) = \sum_{m=0}^\infty \frac{(-\lambda)^m}{m!} \sum_{G \in \mathcal{F}(n,m)} \mathcal{V}(G). \qquad (26)$$

In particular this procedure applies to the case when ν_0 is Gaussian. One recovers the conventional Feynman graphs:

Example 3.11. Let ν_0 be a Gaussian measure. Then the only non−vanishing truncated correlation function is the two point function, or in graphical expression, the only empty inner vertex which does not lead to a zero evaluation of the generalized Feynman graph is   ─○─·
A non−vanishing graph, say of ϕ^2−theory for a two point function, is thus of the kind :

Figure 4

where the second graph in Figure 4 is the conventional Feynman graph which is being evaluated by putting a two point function for each line (conventional Feynman rules). In this situation, the two−point function is also called Euclidean propagator.

4   Wick ordering by avoiding self-contractions

As an example, let us first have a look at one interaction vertex (*inner full vertex*) in ϕ^4−interaction case. In Definition 3.9 the following situations are not excluded :
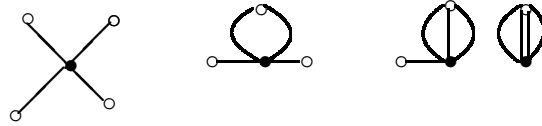

Figure 5

In each of these parts of a generalized Feynman graph we have that one empty vertex is only connected to one and the same interaction vertex. In this situation we say, that a self-contraction occurs at the interaction vertex. Generally we say, that a self-contraction occurs in a generalized Feynman graph, if there exists in this graph, an empty inner vertex connected with one and only one interaction vertex (full inner vertex). Generally speaking, such self-contractions are not problematic for measures ν_0 with $C^\infty(\mathbb{R}^d)$−paths where the truncated Schwinger functions are continuous functions. In the case however where $\langle \phi(x_1), ..., \phi(x_n) \rangle^T_{\nu_0}$ is a function with singularities if $x_j = x_l$, $j \neq l$, a self−contraction leads to a uv-divergence, cf. e.g.

$$\text{(graph)} = \text{(graph)} \cdot \langle \phi^3(y) \rangle^T_{\nu_0} \ . \tag{27}$$

It is thus desirable (even though this often does not resolve the problem of UV divergences completely) to avoid self-contractions. The procedure which is used to do so is called Wick-ordering.
It is known that Wick-ordering can be done in the Gaussian case avoiding self-contractions of the form

no vertices   **Figure 6**

Here we develop a formalism for Wick−ordering in the general case. Because of the one-to-one correspondance between the empty inner vertices of a generalized Feyman graph, and the elements of the related partition, note that a self-contraction in a generalized Feynman graph occurs, if in the related partition I there is a subset I_l that is contained in one of the m sets standing for the interaction vertices and containing four points each. We can thus formulate the problem on the level of partitions and truncated moment functions.

Let $Y = Y(\phi)$ be a " sufficiently integrable " random variable$-Y \in L^p(\nu_0)$ for some $p \geq 1$ would do. As in Equation (12) one has :

$$\langle \phi(u_1)...\phi(u_n)Y \rangle_{\nu_0} = \sum_{I \in \mathcal{P}(\{1,...,n+1\})} \prod_{I_l \in I} \langle I_l \rangle_{\nu_0}^T \tag{28}$$

where we use the convention that on the right hand side we replace $\phi(u_{n+1})$ by $Y(\phi)$.

(28) is to underline the analogy with (12). One can also rewrite (28) in the more explicit form :

$$\langle \phi(u_1)...\phi(u_n)Y \rangle_{\nu_0} = \sum_{I=\{I_1,...,I_k\} \in \mathcal{P}(\{1,...,n\})} \sum_{j=1}^{k} \langle I_j, Y \rangle_{\nu_0}^T \prod_{l=1, \, l \neq j}^{k} \langle I_l \rangle_{\nu_0}^T$$

$$+ \langle Y \rangle_{\nu_0} \sum_{I=\{I_1,...,I_k\} \in P(\{1,...,n\})} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T \tag{29}$$

where $\langle I_j, Y \rangle_{\nu_0}^T = \langle \phi(u_{i_1}), ..., \phi(u_{i_p}), Y \rangle_{\nu_0}^T$ if $I_j = \{i_1, ..., i_p\}$, and $\langle I_l \rangle_{\nu_0}^T = \langle \phi(u_{j_1}), ..., \phi(u_{j_r}) \rangle_{\nu_0}^T$ for $I_l = \{j_1, ..., j_r\}$. It is clear, that there are lots of self−contractions of the type $\langle I_l \rangle_{\nu_0}^T$ on the right hand side of (29). In fact, there is only one single term free of self−contractions: it is $\langle \phi(u_1), ..., \phi(u_n), Y \rangle_{\nu_0}^T$. If Wick−ordering is to remove all self−contractions, one thus has to set :

$$\langle : \phi(u_1)...\phi(u_n) : Y \rangle_{\nu_0} = \langle \phi(u_1), ..., \phi(u_n), Y \rangle_{\nu_0}^T \tag{30}$$

Using (29) and (30) recursively, one obtains :

Definition 4.1. *The Wick−ordering monomial is recursively defined by the equation*

$$: \phi(u_1)...\phi(u_n) := \phi(u_1)...\phi(u_n) - \sum_{\{I_1,...,I_k\} \in \mathcal{P}(\{1,...,n\}) \, k>1} \sum_{j=1}^{k} : I_j : \prod_{l=1, \, l \neq j}^{k} \langle I_l \rangle_{\nu_0}^T$$

$$\tag{31}$$

$$- \sum_{\{I_1,...,I_k\} \in \mathcal{P}(\{1,...,n\})} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T$$

where $: I_j :$ *stands for* $: \phi(u_{i_1})...\phi(u_{i_p}) :$ *if* $I_j = \{i_1, ..., i_p\}$*, and* $\langle I_l \rangle_{\nu_0}^T = \langle \phi(u_{j_1}), ..., \phi(u_{j_r}) \rangle_{\nu_0}^T$ *for* $I_l = \{j_1, ..., j_r\}$*.*

We have to show that this definition in fact solves the problem of self-contractions.

Theorem 4.2. *If in Theorem 3.10 one replaces the interaction ϕ^4 with the interaction $: \phi^4 :$, one obtains the same Feynman graphs and rules with the only difference that all Feynman graphs which have a self−contraction at an interaction vertex are omitted.*

To prove the above theorem, it is sufficient to prove the following more general case :

$$\langle : \phi(y_1)...\phi(y_{p_1}) :: \phi(y_{p_1+1})...\phi(y_{p_1+p_2}) : ... : \phi(y_{p_1+...+p_{m-1}+1})...\phi(y_{p_1+...+p_m}) : \phi(x_1)...\phi(x_n) \rangle_{\nu_0}$$

$$= \sum_{\substack{I \ \in \ \mathcal{P}_{sc}(J_1,...,J_m;Y) \\ I = \{I_1,...,I_k\}}} \prod_{j=1}^{k} \langle I_j \rangle_{\nu_0}^{T} \tag{32}$$

where $J_1, J_2, ..., J_m$ are respectively the subsets $\{1,...,p_1\}$, $\{p_1+1,...,p_1+p_2\}$, ..., $\{p_1+...+p_{m-1}+1,...,p_1+...+p_m\}$, $Y = \{1,...,n\}$ and $\mathcal{P}_{sc}(J_1,...,J_m;Y)$ is the set of all partitions $I = \{I_1,...,I_k\}$ of the elements of $J_1 \cup ... \cup J_m, Y$ such that $I_l \nsubseteq J_q$ for all $1 \leq l \leq k$ and $1 \leq q \leq m$.

Proof of (32):
Let $q = \sum_{i=1}^{m} |J_i|$ where $|J_i|$ is the cardinal number of the set J_i, for example $|J_1| = p_1$. The objective is to prove (32) by induction on q. If $q = 0$, $J_i = \emptyset \ \forall \ i$, then the statement holds simply by definition of truncation.
Let now $q > 0$ and we assume that (32) holds up to $q - 1$. The induction step can be seen as follows: We first apply the definition of Wick ordering to J_m where without loss of generality one may assume $J_m \neq \emptyset$,

$$\langle : J_1 : ... : J_{m-1} :: J_m : Y \rangle_{\nu_0} = \langle : J_1 : ... : J_{m-1} : J_m Y \rangle_{\nu_0} - \langle : J_1 : ... : J_{m-1} :$$

$$\times \left(\sum_{\substack{I=\{I_1,...,I_k\} \ \in \ \mathcal{P}(J_m),\, k > 1}} \sum_{j=1}^{k} : I_j : \prod_{\substack{l \ = \ 1 \\ l \neq j}}^{k} \langle I_l \rangle_{\nu_0}^{T} + \sum_{I=\{I_1,...,I_k\} \ \in \ \mathcal{P}(J_m)} \prod_{j=1}^{k} \langle I_j \rangle_{\nu_0}^{T} \right) Y \rangle_{\nu_0}$$

$$= \langle : J_1 : ... : J_{m-1} : J_m Y \rangle_{\nu_0} - \sum_{\substack{I \ \in \ \mathcal{P}(J_m);\, k \ > \ 1 \\ I = \{I_1,...,I_k\}}} \sum_{j=1}^{k} \langle : J_1 : ... : J_{m-1} :: I_j : Y \rangle_{\nu_0}$$

$$\times \prod_{l=1,\, l \neq j}^{k} \langle I_l \rangle_{\nu_0}^{T} - \langle J_m \rangle_{\nu_0} \langle : J_1 : ... : J_{m-1} : \ Y \rangle_{\nu_0} \tag{33}$$

Using the hypothesis of induction gives :

$$\langle : J_1 : .... : J_{m-1} : J_m Y \rangle_{\nu_0} = \sum_{\substack{I \ \in \ \mathcal{P}_{sc}(J_1,...,J_{m-1};J_m,Y) \\ I = \{I_1,...,I_k\}}} \prod_{j=1}^{k} \langle I_j \rangle_{\nu_0}^{T} \tag{34}$$

because $\sum_{i=1}^{m-1} |J_i| < q$. On the right hand side of (34) there is the sum over all the partitions of the elements of $J_1, ..., J_m, Y$ which do not have self−contractions in $J_1, ..., J_{m-1}$. On the other hand we have

$$
\sum_{I=\{I_1,...,I_k\} \,\in\, \mathcal{P}(J_m),\, k\,>\,1} \sum_{j=1}^{k} \langle : J_1 : ... : J_{m-1} : : I_j : Y \rangle_{\nu_0} \prod_{l=1,\, l\neq j}^{k} \langle I_l \rangle_{\nu_0}^{T}
$$

$$
= \sum_{I=\{I_1,...,I_k\} \,\in\, \mathcal{P}(J_m),\, k\,>\,1} \sum_{j=1}^{k} \sum_{\substack{\tilde{I} \,\in\, \mathcal{P}_{sc}(J_1,...,J_{m-1},I_j;Y) \\ \tilde{I} = \{\tilde{I}_1,...,\tilde{I}_p\}}} \prod_{r=1}^{p} \langle \tilde{I}_r \rangle_{\nu_0}^{T} \prod_{l=1,\, l\neq j}^{k} \langle I_l \rangle_{\nu_0}^{T}
$$

(35)

because $\sum_{i=1}^{m-1} |J_i| + |I_j| < q \; \forall \, j = 1, ..., k$. Using the hypothesis of induction, one obtains

$$
\langle J_m \rangle_{\nu_0} \langle : J_1 : ... : J_{m-1} : Y \rangle_{\nu_0}
$$

$$
= \sum_{I=\{I_1,...,I_k\} \,\in\, \mathcal{P}(J_m)} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^{T} \sum_{\substack{\tilde{I} \,\in\, \mathcal{P}_{sc}(J_1,...,J_{m-1};Y) \\ \tilde{I} = \{\tilde{I}_1,...,\tilde{I}_p\}}} \prod_{r=1}^{p} \langle \tilde{I}_r \rangle_{\nu_0}^{T}
$$

(36)

In (35), the sum is over all the partitions of the elements of $J_1, ..., J_m, Y$ which do not have self−contractions in $J_1, ..., J_{m-1}$ but have self−contractions in J_m and such that the union of the elements of the every partition which are included in J_m, is strictly included in J_m.
(36) is the sum over all the partitions of the elements of $J_1, ..., J_m, Y$ which do not have self−contractions in $J_1, ..., J_{m-1}$ but have self−contractions in J_m and such that the union of the blocks of the every partition which are included in J_m is exactly J_m. Hence :
$\langle : J_1 : ... : J_m : Y \rangle_{\nu_0} = (34) - (35) - (36) =$ the sum over all the partitions of $(J_1, ..., J_m, Y)$ which do not have self−contractions in $J_1, ..., J_m$. ∎
The following clarifies the relation between our graphical notion of Wick ordering and orthogonal decompositions of Wiener-Itô-Segal type:

Theorem 4.3. *The Wick ordered monomials are orthogonal with respect to $L^2(\nu)$ if and only if the measure ν is Gaussian. In particular, Wick ordering in the sense of Definition 4.1 leads to a chaos (orthogonal) decomposition of $L^2(\nu)$ if and only if ν is a Gaussian measure.*

Proof. Coincidence of Wick ordering and the orthogonal decomposition is well-known for Gaussian measures. If ν is not a Gaussian measure, then there exist $u_1, ..., u_r \in \mathbb{R}^d$, $r \geq 3$ such that $\langle \phi(u_1), ..., \phi(u_r) \rangle^T \neq 0$. Let r be the minimal number with this property. Then

$$
\langle : \phi(u_1) :: \phi(u_2)...\phi(u_r) : \rangle_\nu = \langle \phi(u_1), \phi(u_2), ..., \phi(u_r) \rangle_\nu^T.
$$

Hence $\langle : \phi(u_1) :: \phi(u_2)...\phi(u_r) : \rangle_\nu \neq 0$. Hence the Wick ordered monomials are not orthogonal with respect to $L^2(\nu)$. ∎

5 Expansion of the free energy into connected Feynman graphs

The topic of this section is a classical problem of statistical mechanics, namely the expansion of the free energy $\ln \Xi_\Lambda(\lambda)$ of a statistical mechanics system into connected Feynman graphs. Here we give a new and measure independent proof of a very general linked cluster theorem:

Theorem 5.1. *Only connected generalized Feynman graphs contribute to the perturbation expansion of the free energy, i.e., in the sense of formal power series in λ*

$$\ln \Xi_\Lambda(\lambda) = \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \sum_{\substack{G \in \mathcal{F}(0,m) \\ G \text{ connected}}} \mathcal{V}_\Lambda(G) \tag{37}$$

where $\mathcal{V}_\Lambda(G)$ is the value associated with G according to the Feynman rules given in Definition 3.9

The remainder of this section is dedicated to the proof of this theorem. We would like to remark that this theorem also holds when the interaction is Wick ordered, as the proof we give here carries over word by word.

Let us first observe that $\Xi_\Lambda(\lambda)$ is the Laplace transform of the random variable V_Λ, hence it is the generating functional for the moments of V_Λ. It follows ¿from the basic linked cluster theorem 3.3 (that is valid also for the Laplace transform) that $\ln \Xi_\Lambda(\lambda)$ is the generating functional of the truncated moments of V_Λ, i.e.

$$\ln \Xi_\Lambda(\lambda) = \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \langle \underbrace{V_\Lambda, \ldots, V_\Lambda}_{m \text{ times}} \rangle_{\nu_0}^T. \tag{38}$$

Furthermore, using induction and Fubini's theorem, it is easy to show that

$$\langle \underbrace{V_\Lambda, \ldots, V_\Lambda}_{m \text{ times}} \rangle_{\nu_0}^T = \int_{\Lambda^m} \langle \phi^4(y_1), \cdots, \phi^4(y_m) \rangle_{\nu_0}^T dy_1 \cdots dy_m \tag{39}$$

The following theorem is the main technical step. It expands the truncated moments $\langle J_1, ..., J_n \rangle_{\nu_0}^{(T)}$ into truncated moments $\langle \phi(u_1), ..., \phi(u_m) \rangle_{\nu_0}^T$:

Theorem 5.2. *The following identity holds:*

$$\langle J_1, ..., J_n \rangle_{\nu_0}^T = \sum_{I=\{ I_1,...,I_k\} \in P_c(\bigcup_{l=1}^n J_l)} \prod_{l=1}^{k} \langle I_l \rangle_{\nu_0}^T \tag{40}$$

where $P_c(\bigcup_{l=1}^n J_l)$ is the all elements $I = \{ I_1, ..., I_k\} \in P(\bigcup_{l=1}^n J_l)$ such that $\forall\, 1 \le J_{j_1} < \ ... < J_{j_l} \le n$ with $l < n$, and $\forall\, 1 \le i_1 < ... < i_r \le k$, one has $\bigcup_{q=1}^l J_{j_q} \ne \bigcup_{p=1}^r I_{i_p}$.

Proof. To prove (40) it is sufficient to prove that the right hand side of (40) fulfills the defining equation (12), i.e.,

$$\langle J_1...J_n \rangle_{\nu_0} = \sum_{I=\{I_1,...,I_k\} \in P(\{1,...,n\})} \prod_{l=1}^{k} \sum_{\substack{Q^l \in\ P_c(\bigcup_{q \in I_l} J_q) \\ Q^l = \{Q_1^l,...,Q_{r_l}^l\}}} \prod_{a=1}^{r_l} \langle Q_a^l \rangle_{\nu_0}^T \tag{41}$$

for any collection $J_1, ..., J_n$. Expanding the left hand side into truncated functions, we get

$$
\sum_{\substack{\alpha \in P(\bigcup_{l=1}^{n} J_l) \\ \alpha = \{\alpha_1, ..., \alpha_k\}}} \prod_{j=1}^{k} \langle \alpha_j \rangle_{\nu_0}^{T} = \sum_{\substack{I \in P(\{1, ..., n\}) \\ I = \{I_1, ..., I_k\}}} \sum_{\substack{Q^1 \in P_c(\bigcup_{q \in I_1} J_q) \\ Q^1 = \{Q_1^1, ..., Q_{r_1}^1\}}} \prod_{a=1}^{r_1} \langle Q_a^1 \rangle_{\nu_0}^{T} \cdots
$$

$$
\cdots \sum_{\substack{Q^k \in P_c(\bigcup_{q \in I_k} J_q) \\ Q^k = \{Q_1^k, ..., Q_{r_k}^k\}}} \prod_{a=1}^{r_k} \langle Q_a^k \rangle_{\nu_0}^{T}
$$

$$
= \sum_{\substack{I \in P(\{1, ..., n\}) \\ I = \{I_1, ..., I_k\}}} \sum_{\substack{Q^1 \in P_c(\bigcup_{q \in I_1} J_q) \\ Q^1 = \{Q_1^1, ..., Q_{r_1}^1\}}} \cdots \sum_{\substack{Q^k \in P_c(\bigcup_{q \in I_k} J_q) \\ Q^k = \{Q_1^k, ..., Q_{r_k}^k\}}}
$$

$$
\prod_{a=1}^{r_1} \langle Q_a^1 \rangle_{\nu_0}^{T} \quad \cdots \quad \prod_{a=1}^{r_k} \langle Q_a^k \rangle_{\nu_0}^{T} \tag{42}
$$

Let us focus on the case $\sharp J_l = 4$, for simplicity (these considerations however carry over to the general case). Then there is a one to one correspondence between generalized Feynman graphs with ϕ^4-interaction vertices and partitions of $J_1, \ldots, J_m$, cf. section 3. Following the prescription of the equivalence between partitions and Feynman graphs, it is not difficult to see that the associated generalized Feynman graphs to $P_c(\bigcup_{l=1}^{n} J_l)$ are just the connected Feynman generalized graphs. Hence, the first sum on the right hand side of (42) is a sum over all possible connected components of the graph with "vertices" $J_1, \ldots, J_m$. The remaining sums then give all possibilities of graphs which, have exactly the given connected components. Hence on both sides of (42) we get a sum over all generalized Feynman graphs or, equivalently, over all partitions. ∎

As explained in the proof above, the interpretation of the condition in the sum of (40) just means that the associated Feynman graph is connected. Combining thus the equations (38)–(40) one obtains Theorem 5.1.

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